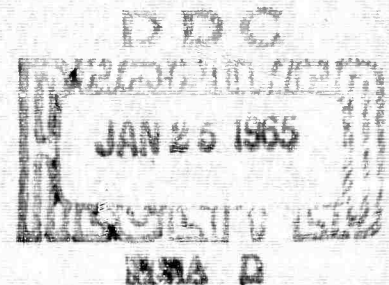MEMORANDUM
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A NEW DERIVATION OF THE INTEGRO-DIFFERENTIAL EQUATIONS FOR CHANDRASEKHAR'S X AND Y FUNCTIONS

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SUMMARY AND PREFACE

The X and Y functions of Chandrasekhar are of great importance
in the theory of multiple scattering in a finite slab. Their properties
are best determined analytically through the use of integral equations.
From the computational view, though, they are best treated as solutions
of a system of integro-differential equations.

Our aim is to produce a derivation from first principles which
illuminates the nature of the integro-differential equations and which
will aid us in our computational studies of transport processes. A
knowledge of integral equations and Chandrasekhar's book, Radiative
Transfer,[1] is assumed.

I. INTRODUCTION

In a preceding paper,[2] with the aid of the invariant imbedding technique, we derived the functional equations for the Fredholm resolvent for the photon diffusion equation in the case of the slab geometry. In the present paper, making use of the relationship between the source function and the resolvent obtained previously, we show a new approach for finding the integro-differential equations for the X and Y functions of Chandrasekhar[1] in the theory of radiative transfer in a finite flat layer. Since these equations are useful for the numerical calculation of the X and Y functions, they play a significant role in the inverse problem of light scattering in slab geometry.[3] Let us also note that there is no problem of uniqueness in the case of conservative scattering when we employ the integro-differential equations for the X and Y functions rather than the integral equations.

II. MILNE'S INTEGRAL EQUATION

Consider the diffuse reflection and transmission of light by a plane-arallel, homogeneous, and isotropically scattering atmosphere of the finite optical thickness $\tau_1 > 0$ with the distribution $B_1(\tau)$ of internal emission sources. Let a parallel beam of radiation of constant net flux πF per unit area normal to the direction of propagation be incident on the surface $\tau = 0$ in a fixed angle $\cos^{-1}\mu_0 (0 < \mu_0 \leq 1)$ with the inwards normal.

Following the notation of Chandrasekhar,[1] we shall use $I(\tau, + \mu)$ $(0 < \mu \leq 1)$ to denote the intensity of radiation at level

in the direction $+\mu$ directed towards the top surface $\tau = 0$, and further $I(\tau,-\mu)(0 < \mu \leq 1)$ for that directed towards the bottom surface $\tau = \tau_1$.

The equation of transfer in the diffuse radiation field takes the form

$$\mu \frac{dI(\tau,\mu)}{d\tau} = I(\tau,\mu) - \frac{1}{2} \overline{\lambda} \int_{-1}^{+1} I(\tau,\mu')d\mu' - B(\tau) , \qquad (1)$$

where $\overline{\lambda}$ is the albedo for single scattering and $B(\tau)$ is

$$B(\tau) = B_1(\tau) + \frac{1}{4} \overline{\lambda} F e^{-\tau/\mu_0} . \qquad (2)$$

Equation (1) should be solved subject to the boundary conditions

$$I(0,-\mu) = 0 \quad \text{and} \quad I(\tau_1,+\mu) = 0 \quad (0 < \mu \leq 1) . \qquad (3)$$

Equation (3) represents the fact that no diffuse radiation is incident on the surfaces from the outside.

Starting from the formal solution of Eq. (1), we have the Milne integral equation for the source function $S(\tau)$

$$S(\tau) = \overline{\lambda} \overline{\Lambda}_\tau \{S(t)\} + B(\tau) , \qquad (4)$$

where the truncated Hopf operator $\overline{\Lambda}$ is given by

$$\overline{\Lambda}_\tau \{f(t)\} = \frac{1}{2} \int_0^{\tau_1} f(t)E_1(|t-\tau|)dt \qquad (0 \leq \tau \leq \tau_1) . \qquad (5)$$

In Eq. (5) E_1 is the first exponential integral

$$E_1(\tau) = \int_0^1 e^{-\tau/\mu} \frac{d\mu}{\mu} .$$

Alternatively, Eq. (4) is written in the form

$$S(\tau) = B(\tau) + \int_0^{\tau_1} K(\tau,t;\tau_1)B(t)dt , \qquad (6)$$

where the Fredholm resolvent $K(\tau,t;\tau_1)d\tau dt$ is the probability that a photon emitted in the optical depth interval $(t,t+dt)$ will be re-emitted in the optical depth interval $(\tau,\tau+d\tau)$ after one or more scattering processes.

Then, once the resolvent K has been determined, Eq. (6) enables us to evaluate the source function which gives us the intensities of the emergent diffuse radiation as follows:

$$I(0,+\mu) = \int_0^{\tau_1} S(t)e^{-t/\mu} \frac{dt}{\mu} , \qquad (7)$$

$$I(\tau_1,-\mu) = \int_0^{\tau_1} S(t)e^{-(\tau_1-t)/\mu} \frac{dt}{\mu} . \qquad (8)$$

The directly transmitted intensity is $\frac{F}{2} e^{-\tau_1/\mu_0} \delta(\mu-\mu_0)$.

III. FREDHOLM RESOLVENT OF THE PHOTON DIFFUSION EQUATION

In a preceding paper,[2] with the aid of the invariant imbedding technique, we obtained the following equations (making use of the optical homogeneous properties of the medium):

$$\frac{\partial K}{\partial \tau_1}(x,y;\tau_1) = K(\tau_1-x,0;\tau_1)K(0,\tau_1-y;\tau_1) , \qquad (9)$$

$$\frac{\partial K}{\partial x}(x,y;\tau_1) + \frac{\partial K}{\partial y} = K(x,0;\tau_1)K(0,y;\tau_1) - K(\tau_1-x,0;\tau_1)K(0,\tau_1-y;\tau_1). \qquad (10)$$

Then,

$$K(x,y;\tau_1) = K(x-y,0;\tau_1) + \int_0^y \Big[K(t+x-y,0;\tau_1)K(0,t;\tau_1)$$

$$- K(\tau_1-t-x+y,0;\tau_1)K(0,\tau_1-t;\tau_1)\Big]dt \ , \tag{11}$$

where $x > y$.

With the aid of Eq. (11), the resolvent $K(x,y;\tau_1)$ is expressed in terms of $K(x,0;\tau_1)$, a significant reduction. Equations (9), (10), and (11) reduce to those given by Sobolev.[4]

On the other hand, by means of another analytical procedure,[5] we know that

$$K(x,y;\tau_1) = k(x,y) + \int_0^{\tau_1} K(x,y';\tau_1)k(y',y)dy' \tag{12}$$

where

$$k(y',y) = \frac{1}{2}\,\bar{\lambda}\,E_1(|y'-y|) \ . \tag{13}$$

Furthermore, from Eq. (4) with $B_1(\tau) = 0$ and $F = 4$, we have[2]

$$\phi(x,\tau_1) = g(x) + \int_0^{\tau_1} \phi(x',\tau_1)k(x',x)dx' \ , \tag{14}$$

where $k(x,x')$ is given by Eq. (13), and

$$\phi(x,\tau_1) = \frac{1}{2}\int_0^1 S^*(x,\mu)\,\frac{d\mu}{\mu} \ , \tag{15}$$

$$g(x) = \frac{1}{2}\,\bar{\lambda}\,E_1(x) \ . \tag{16}$$

In Eq. (15) the source function $S^*(x,\mu)$ satisfies the auxiliary equation

$$[1 - \bar{\lambda}\, \bar{\Lambda}]_\tau \{S^*(t,\mu)\} = \bar{\lambda}\, e^{-\tau/\mu} \, , \tag{17}$$

where 1 on the left-hand side represents an identity operator. The quantity $S^*(\tau,\mu)$ corresponds to the probability that a photon absorbed at level τ will reappear in the direction $+\mu (0 < \mu \leq 1)$ in the radiation emerging from the surface $\tau = 0$. Equation (17) is also called a photon diffusion equation.

In a preceding paper,[2] it is also shown that

$$K(x,0;\tau_1) = \Phi(x,\tau_1) \, , \quad K(\tau_1-x,0;\tau_1) = \Phi(\tau_1-x,\tau_1) \, . \tag{18}$$

IV. THE INTEGRO-DIFFERENTIAL EQUATIONS FOR THE X AND Y FUNCTIONS

On differentiating Eq. (6) with respect to τ_1, we have

$$\frac{\partial S(\tau)}{\partial \tau_1} = K(\tau,\tau_1;\tau_1)B(\tau_1) + \int_0^{\tau_1} \frac{\partial K}{\partial \tau_1}(\tau,t;\tau_1)B(t)dt \, . \tag{19}$$

Making use of Eqs. (9) and (18), Eq. (19) becomes

$$\begin{aligned}
\frac{\partial S(\tau)}{\partial \tau_1} &= \Phi(\tau_1-\tau,\tau_1)B(\tau_1) + \int_0^{\tau_1} \Phi(\tau_1-\tau,\tau_1)\Phi(\tau_1-t,\tau_1)B(t)dt \\
&= \Phi(\tau_1-\tau,\tau_1)\left[B(\tau_1) + \int_0^{\tau_1} \Phi(\tau_1-t,\tau_1)B(t)dt\right] \\
&= \Phi(\tau_1-\tau,\tau_1)S(\tau_1) \\
&= \frac{1}{2} S(\tau_1) \int_0^1 S^*(\tau_1-\tau,\mu) \frac{d\mu}{\mu} \, .
\end{aligned} \tag{20}$$

Putting

$$B_1(\tau) = 0 \qquad \text{and} \qquad F = 4 \ , \tag{21}$$

Eq. (20) for $S^*(\tau,\mu)$ becomes

$$\frac{\partial S^*(\tau,\mu)}{\partial \tau_1} = \frac{1}{2} \ S^*(\tau_1,\mu) \int\limits_0^1 S^*(\tau_1-\tau,\mu') \ \frac{d\mu'}{\mu'} \ , \tag{22}$$

which reduces to that given by Sobolev,[6] Busbridge[7] and Ueno.[8]

Writing

$$S^*(0,\mu) = \overline{\lambda} \ X(\mu,\tau_1), \qquad S^*(\tau_1,\mu) = \overline{\lambda} \ Y(\mu,\tau_1) \ , \tag{23}$$

and letting $\tau = 0$, from Eq. (22) we find the desired equation

$$\frac{\partial X(\mu,\tau_1)}{\partial \tau_1} = \frac{1}{2} \ \overline{\lambda} \ Y(\mu,\tau_1) \int\limits_0^1 Y(\mu',\tau_1) \ \frac{d\mu'}{\mu'} \ . \tag{24}$$

Equation (24) is one of the requisite integro-differential equations for the X and Y functions.

Now we derive the other. Let the Milne integral equation for the source function $S(\tau_1-\tau)$ be

$$S(\tau_1-\tau) = \overline{\lambda} \ \overline{\Lambda}_\tau \{S(\tau_1-t\} + B(\tau_1-\tau) \ . \tag{25}$$

With the aid of the Fredholm resolvent, Eq. (25) is written in the form

$$S(\tau_1-\tau) = B(\tau_1-\tau) + \int\limits_0^{\tau_1} \overline{K}(\tau,t;\tau_1)B(\tau_1-t)dt. \tag{26}$$

On differentiating Eq. (26) with respect to τ_1, we get

$$\frac{\partial S(\tau_1 - \tau)}{\partial \tau_1} = \frac{\partial B(\tau_1 - \tau)}{\partial \tau_1} + K(\tau, \tau_1; \tau_1) B(0) + \int_0^{\tau_1} \frac{\partial K(\tau, t; \tau_1)}{\partial \tau_1} B(\tau_1 - t) dt$$

$$+ \int_0^{\tau_1} K(\tau, t; \tau_1) \frac{dB(\tau_1 - t)}{d\tau_1} dt \ . \tag{27}$$

Making use of Eq. (21), from Eq. (25) we have the auxiliary equation for the source function $S^*(\tau_1 - \tau, \mu)$

$$[1 - \bar{\lambda} \bar{\Lambda}]_\tau \{ S^*(\tau_1 - t, \mu) \} = \bar{\lambda} \, e^{-(\tau_1 - \tau)/\mu} \ . \tag{28}$$

The combinations of Eqs. (27) and (28) leads to

$$\frac{\partial S^*(\tau_1 - \tau, \mu)}{\partial \tau_1} = -\frac{1}{\mu} \bar{\lambda} \, e^{-(\tau_1 - \tau)/\mu} + K(\tau, \tau_1; \tau_1) \bar{\lambda}$$

$$+ \bar{\lambda} \int_0^{\tau_1} \frac{\partial K(\tau, t; \tau_1)}{\partial \tau_1} e^{-(\tau_1 - t)/\mu} dt - \frac{\bar{\lambda}}{\mu} \int_0^{\tau_1} K(\tau, t; \tau_1) e^{-(\tau_1 - t)/\mu} dt$$

$$= -\frac{1}{\mu} \left[\bar{\lambda} \, e^{-(\tau_1 - \tau)/\mu} + \bar{\lambda} \int_0^{\tau_1} K(\tau, t; \tau_1) e^{-(\tau_1 - t)/\mu} dt \right] \tag{29}$$

$$+ \bar{\lambda} \left[K(\tau_1 - \tau, 0; \tau_1) + \int_0^{\tau_1} \frac{\partial K(\tau, t; \tau_1)}{\partial \tau_1} e^{-(\tau_1 - t)/\mu} dt \right]$$

$$= -\frac{1}{\mu} S^*(\tau_1 - \tau, \mu) + \frac{1}{2} S^*(0, \mu) \int_0^1 S^*(\tau_1 - \tau, \mu') \frac{d\mu'}{\mu'} \ .$$

Then, letting $\tau = 0$ in Eq. (29) and recalling Eq. (23), we have

$$\frac{\partial Y(\mu, \tau_1)}{\partial \tau_1} = -\frac{1}{\mu} Y(\mu, \tau_1) + \frac{1}{2} \bar{\lambda} \, X(\mu, \tau_1) \int_0^1 Y(\mu', \tau_1) \frac{d\mu'}{\mu'} \ . \tag{30}$$

Equation (30) is the required integral equation for the X and Y functions. Equation (29) reduces to that given by Busbridge[9] and Ueno,[10] respectively.

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